Differentiability and differentiability in a piecewise function

Given the function

$$f(x,y) = \begin{cases} \dfrac{2x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

1. Analyze the continuity at the origin
2. Analyze the differentiability at the origin
3. Is it differentiable at the origin? Justify

Solutions

1. The function exists at the origin and equals 0, now it is only necessary to see if the limit exists at the origin, for that we take the parabolic limit: $y = mx^2$

$$\lim_{x \to 0} \frac{2x^2mx^2}{x^4 + m^2x^4} = \lim_{x \to 0} x^4 \frac{2m}{x^4(1 + m^2)} = \frac{2m}{1 + m^2}$$

As it depends on m we can say that the limit does not exist. And therefore, the function is not continuous.

2. We differentiate by definition:

$$f'_x = \lim_{h \to 0} \frac{\frac{2(0+h)^2 * 0}{(0+h)^4 + 0^2} - 0}{h} = \lim_{h \to 0} \frac{0/h^4}{h} = 0$$

$$f'_y = \lim_{h \to 0} \frac{\frac{2(0)^2 * (0+h))}{(0)^4 + (0+h)^2} - 0}{h} = \lim_{h \to 0} \frac{0/h^2}{h} = 0$$

3. **The necessary condition for the function to be differentiable is that it is continuous at the point and has partial derivatives at the point. Since the function is not continuous, it is not differentiable.**